Practical session: Linear SVM for two class separable data

Stéphane Canu
scanu@insa-rouen.fr, asi.insa-rouen.fr/~scanu

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Practical session description

This practical session aims at writing two functions solving the separable two classes classification problem with linear Support Vector Machines (SVM) as a quadratic program in different situations: primal dual, without and with noise. To make it work, you are supposed to have CVX installed (you can download it from http://cvxr.com/cvx/) as well as the quadprog matlab function available in the matlab optimization toolbox.

Ex. 1 — Linear SVM for two class separable data

1. Generate a set of 100 data points in dimension 2, uniformly distributed in the square (0,4). To make this data set linearly separable, set the labels to 1 for the points above the separating line $w^T x + b = 0$ with $w = (4, -1)$ and $b = -6$. This will be your training set.

   ```matlab
   n = 100; % sample size up to 200000 !
   rand('seed',2); % fix the randomness
   Xi = 4*rand(n,2); % build the training set
   q = 0;
   Xi = [Xi 4*rand(n,q)];
   [n,p] = size(Xi);
   bt = -6; % define the separation line bias
   wt = [4 ; -1]; % define the separation line vector
   yi = sign(wt(1) * Xi(:,1) + wt(2) * Xi(:,2) + bt);
   ```

2. Plot the training set, using red circle for class 1 data points and blue circles for the others. Draw separating line $w^T x + b = 0$ with $w = (4, -1)$ and $b = -6$ (in green to get figure 1).

   ```matlab
   plot(Xi(:,1),Xi(:,2),’or’);
   hold on
   plot(Xi(find(yi==1),1),Xi(find(yi==1),2),’ob’);
   x1 = 0;
   y1 = (-bt-(wt(1)*x1))/wt(2);
   x2 = 4;
   y2 = (-bt-(wt(1)*x2))/wt(2);
   plot([x1 x2],[y1 y2],’g’,’LineWidth’,2)
   ```

Figure 1: result of TP 1
3. Max margin SVM
   a) Using CVX, give a matlab code for solving
   \[
   \begin{aligned}
   \max_{m,v,a} & \quad m \\
   \text{subject to} & \quad y_i(v^\top x_i + a) \geq m; \quad i = 1, n \\
   & \quad \|v\|^2 = 1
   \end{aligned}
   \]
   ```matlab
   cvx_begin
   variables v(p) a m
   maximize( m )
   subject to
   yi.*(Xi*v + a) >= m;
   v'*v <= 1;
   cvx_end
   ```
   b) How long does it take? (use tic/toc matlab instructions)
   c) Find the indices of the support vectors
   ```matlab
   vec_sup = find(yi.*(Xi*v + a) <= m+eps^.3);
   ```
   d) Draw the separating line found by the max margin SVM and the associated margin and support vectors
   ```matlab
   x1 = 0;
   y1 = (-a-(v(1)*x1))/v(2);
   z1 = (m-a-(v(1)*x1))/v(2);
   zm1 = (-m-a-(v(1)*x1))/v(2);
   x2 = 4;
   y2 = (-a-(v(1)*x2))/v(2);
   z2 = (m-a-(v(1)*x2))/v(2);
   zm2 = (-m-a-(v(1)*x2))/v(2);
   plot([x1 x2],[y1 y2],’r’)
   plot([x1 x2],[z1 z2],’r’)
   plot([x1 x2],[zm1 zm2],’r’)
   plot(Xi(vec_sup,1),Xi(vec_sup,2),’sm’,’MarkerSize’,10);
   ```

4. Linear SVM minimizing the norm (usual form)
   a) Using CVX, give a matlab code for solving
   \[
   \begin{aligned}
   \min_{w,b} & \quad \frac{1}{2}\|w\|^2 \\
   \text{subject to} & \quad y_i(w^\top x_i + b) \geq 1; \quad i = 1, n
   \end{aligned}
   \]
   ```matlab
   cvx_begin
   variables w(p) b
   dual variables pi
   minimize(.5*w'*w)
   subject to
   pi : yi.*(Xi*w + b) >= 1;
   cvx_end
   ```
   b) Check that the results given by the max margin and the min norm SVM are the same i.e.
   \[
   v = \frac{w}{\|w\|}; \quad v = mw \quad \text{and} \quad a = \frac{b}{\|w\|}, a = mb
   \]
   ```matlab
   [v w/norm(w) w/m]
   [a b/norm(w) b a/m]
   ```
5. Write the KKT condition associated with the solution
   a) Based on the previous results \((w, b)\), retrieve the active set (the indices of support vectors)

   \[
   A = \text{find}(y_i .*(X_i*w + b) == 1);
   \]
   \[
   A = \text{find}(y_i .*(X_i*w + b) <= 1.00001);
   \]
   \[
   cA = \text{length}(A);
   \]

   b) Write the KKT system of equation

   \[
   DA = \text{diag}(y_i(A));
   \]
   \[
   KKT = \begin{bmatrix}
   \text{eye}(p) & -X_i(A,:) ' \text{DA} & \text{zeros}(p,1) \\
   -DA* X_i(A,:) & \text{zeros}(cA) & -y_i(A) \\
   \text{zeros}(1,p) & -y_i(A)' & \text{zeros}(1)
   \end{bmatrix};
   \]
   \[
   Kb = \begin{bmatrix}
   \text{zeros}(p,1) \\
   -\text{ones}(cA,1) \\
   0
   \end{bmatrix};
   \]
   \[
   sol = KKT \backslash Kb;
   \]

   c) Check that the solution provided by matlab and the one given by solving the KKT are the same

   \[
   \begin{bmatrix}
   w; pi(A); b
   \end{bmatrix} \text{ sol}
   \]

6. SVM and quadratic programming
   a) Rewrite the min norm SVM problem as a quadratic program in its standard form and use `quadprog` or `cplexqp` to solve it

   \[
   \begin{bmatrix}
   H = \text{eye}(p); \\
   H(p+1,p+1) = 0; \\
   f = \text{zeros}(p+1,1); \\
   A = -[\text{diag}(y)*X_i y_i ]; \\
   bb = -\text{ones}(n,1);
   \end{bmatrix}
   \]
   \[
   x = \text{quadprog}(H,f,A,bb);
   \]

   b) Check that the results provided by CVX and `quadprog` are the same

   \[
   [x \ [w; b]]
   \]

   c) How long does it take. Is it slower or faster than CVX (and why)?

7. Max Margin SVM in the dual
   a) Using CVX, give a matlab code for solving Max Margin SVM in the dual

   \[
   \begin{align*}
   \min_{\alpha} & \quad \frac{1}{2} \alpha^\top G \alpha - \sum_{i} \alpha_i \\
   \text{with} & \quad \alpha^\top y_i = 0; \\
   & \quad 0 \leq \alpha_i; \quad i = 1,n
   \end{align*}
   \]

   \[
   G = (y_i*y_i') .*(X_i* X_i'); \\
   e = \text{ones}(n,1);
   \]

   \[
   \text{cvx_begin}
   \begin{align*}
   \text{variable} & \quad a(n) \\
   \text{dual variables} & \quad de \ dp \\
   \text{minimize} & \quad (1/2*a'*G*a - e'*a) \\
   \text{subject to} & \quad y_i'*a == 0; \\
   & \quad dp : \ a >= 0;
   \end{align*}
   \]

   \[
   \text{cvx_end}
   \]

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b) Check that the dual variable of the primal are the same as the variables of the dual

\[ \{a, pi\} \]

c) Check that the dual variable of the primal are the same as the variables of the dual

\[ \{b, de\} \]

d) Using the representer theorem (KKT for stationarity with respect to \(w\)), recompute \(w\) using the dual variables

\[ \{w, Xi'*(yi*a)\} \]

8. Using `quadprog` to solve both primal and dual SVM formulations

a) Modify the outputs of the `quadprog` you wrote for solving the min norm SVM problem to get the dual variables (the Lagrange multipliers)

```
% \[X,FVAL,EXITFLAG,OUTPUT,LAMBDA\] = QUADPROG(H,f,A,b) returns the set
%  Lagrangian multipliers LAMBDA, at the solution: LAMBDA.ineqlin %
%  linear inequalities A, LAMBDA.eqlin for the linear equalities Aeq %
%  LAMBDA.lower for LB, and LAMBDA.upper for UB.H = [eye(p)];

H(p+1,p+1) = 0;
f = zeros(p+1,1);
A = -(diag(yi)*Xi*yi);
bb = -ones(n,1);
[xp, VAL,EXITFLAG,OUTPUT,lambda] = quadprog(H,f,A,bb);
```

b) Rewrite the min norm SVM dual problem as a quadratic program in its stand at form and use `quadprog` or `cplexqp` to solve it

```
l = eps^.5;
G = G + l*eye(n); % 7) the secret to make it work
tic
ad = quadprog(G,-e,[],[],yi',0,zeros(n,1),inf*ones(n,1));
```

c) Download and instal the SVMKM toolbox from


Solve the same min norm SVM dual problem using the `monqp` solver included in the SVMKM toolbox.

```
% function \[xnew, lambda, pos\] = monqp(H,c,A,b,C,l, verbose,X,ps,xinit)
%  \% min 1/2 x' H x - c' x
%  \% x
%  \% contrainte A' x = b
%  \% et 0 <= x_i <= C_i
%  \[alpha, b, pos\] = monqp(G,e,yi,0,inf,l,0);

[aqp, pos] = alpha;
wqp = Xi(pos,:)'*(yi(pos).*alpha);
```

d) Using the output of `monqp`, recompute the whole dual variables and the associated primal variables.

9. Compare all the results and computing time.

10. Write two matlab functions `SVMClass`, `SVMVal` for solving the separable two classes classification problem with linear Support Vector Machines (SVM) in the primal as a quadratic program.

```
[w,b] = SVMClass(Xi,yi,opt);
% opt for some options
% you may also ofer the possibility for the user too choose the solver
[y_pred] = SVMVal(Xtest,w,b);
```